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Noncommutative chiral gauge theories on the lattice with manifest star-gauge invariance

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ABSTRACT: We show that noncommutative U(r) gauge theories with a chiral fermion in the adjoint representation can be constructed on the lattice with manifest star-gauge invariance in arbitrary even dimensions. Chiral fermions are implemented using a Dirac operator which satisfies the Ginsparg-Wilson relation. A gauge-invariant integration measure for the fermion fields can be given explicitly, which simplifies the construction as compared with lattice chiral gauge theories in ordinary (commutative) space-time. Our construction includes the cases where continuum calculations yield a gauge anomaly. This reveals a certain regularization dependence, which is reminiscent of parity anomaly in commutative space-time with odd dimensions. We speculate that the gauge anomaly obtained in the continuum calculations in the present cases can be cancelled by an appropriate counterterm.

KEYWORDS: Non-Commutative Geometry, Anomalies in Field and String Theories, Lattice Gauge Field Theories, Matrix Models.

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1. Introduction

Recently gauge theories on noncommutative space-time have attracted much attention because of their intimate relationships to string theories (for a review see [1]). In particular, Matrix Theory [2] and the IIB matrix model [3], which are conjectured to provide nonperturbative definitions of string/M theories, give rise to noncommutative gauge theory on toroidal compactifications [4]. The particular noncommutative toroidal compactification is interpreted as being the result of the presence of a background Neveu-Schwarz two-form field, and it can also be understood in the context of open string quantization in D-brane backgrounds [5]. Furthermore, in ref. [6] it has been shown that the IIB matrix model with D-brane backgrounds is described by noncommutative gauge theory. Based on these developments, a lattice formulation of noncommutative gauge theory has been established [7]–[10]. (see also [11, 12] for reviews.) In this paper, we apply this lattice formulation to studying chiral fermions in noncommutative space-time.

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Construction of chiral gauge theories on the lattice was considered to be difficult for a long time due to the well-known no-go theorem [13]. The situation has changed drastically since a manifestly gauge-invariant construction of abelian lattice chiral gauge theories has been established [14]. The no-go theorem is circumvented by requiring the Dirac action on the lattice (without species doublers) to be manifestly invariant under a *modified* chiral transformation [15], which reduces to the usual chiral transformation only in the continuum limit. Such a Dirac action can be constructed by the use of a Dirac operator [16] which satisfies the Ginsparg-Wilson relation [17]. The Dirac operator breaks ultra-locality, but it still preserves locality in the sense that the corresponding kernel decays exponentially at long distances [18]. In this formalism the construction of chiral gauge theories boils down to the question of choosing an integration measure for the chiral fermion fields in a gauge invariant way.[1]

We take the same approach in the noncommutative space-time with arbitrary even dimensions. When the chiral fermion transforms as the adjoint representation under star-gauge transformations, we can construct a star-gauge invariant fermion measure *explicitly* by exploiting a peculiar property of noncommutative geometry. This is a considerable simplification compared with the construction of abelian lattice chiral gauge theories in commutative space-time [14], where only the *existence* of a gauge-invariant fermion measure is known nonperturbatively. Moreover, our construction can be extended to the case with higher-rank gauge groups in a straightforward manner.

Noncommutative chiral gauge theories have been also studied in the continuum by various regularizations. In ref. [20] the gauge anomaly has been calculated using a variant of Fujikawa's method [21, 22]. It was found there that four-dimensional noncommutative chiral gauge theories with fermions in the fundamental representation of $\mathrm{U}(r)$ are anomalous. The result was confirmed from the viewpoint of descent equations [23], and also reproduced diagrammatically using dimensional regularization [24]. On the other hand, if the chiral fermions transform in the adjoint representation of the gauge group, the total anomaly cancels in four dimensions and gauge invariance is preserved. This is not so suprising since four-dimensional chiral fermions in the adjoint representation can be formulated alternatively in terms of Majorana fermions. However this is not the case in two dimensions, where we find that a similar calculation yields a nonvanishing gauge anomaly. Comparing these results with our lattice construction, we observe a certain regularization dependence, which is reminiscent of the parity anomaly in commutative space-time with an odd number of dimensions (See refs. [25] and references therein). We speculate that the gauge anomaly obtained in the continuum calculations might be cancelled by an appropriate counterterm in these cases.

[1] While this paper was being prepared, we received a preprint [19] where a different type of manifestly gauge invariant construction of lattice chiral gauge theories has been proposed.

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The rest of the paper is organized as follows. In section 2, we review the lattice formulation of noncommutative gauge theories. In section 3, we introduce chiral fermion in noncommutative gauge theories. In section 4, we present some calculations of the gauge anomaly in noncommutative chiral gauge theories in the continuum. Finally, section 5 is devoted to summary and discussions.

2. Lattice formulation of noncommutative gauge theories

In this section, we briefly review the lattice formulation of noncommutative gauge theory [8, 9, 10]. For concreteness, we shall work with a specific example based on twisted Eguchi-Kawai model [26, 27].

2.1 Discrete noncommutative geometry

The starting point of noncommutative geometry (on $\mathbb{R}^D$) is to replace the space-time coordinates by hermitian operators obeying the commutation relation

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\theta_{\mu\nu}\,, \tag{2.1}$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ are real-valued c-numbers with dimensions of length squared. One may also introduce an anti-hermitian derivation $\hat{\partial}_\mu$ through the commutation relation

$$\left[\hat{\partial}_\mu\,,\,\hat{x}_\nu\right] = \delta_{\mu\nu}\,, \qquad \left[\hat{\partial}_\mu\,,\,\hat{\partial}_\nu\right] = i\,c_{\mu\nu}\,, \tag{2.2}$$

where $c_{\mu\nu} = -c_{\nu\mu}$ are real-valued c-numbers. Let us introduce a D-dimensional discretized torus defined by[2]

$$\Lambda_\ell = \left\{(x_1, x_2, \dots, x_D)\,|\,x_\mu = \epsilon n_\mu\,,\, n_\mu \in \mathbb{Z}\,,\, -\frac{L-1}{2} \leq n_\mu \leq \frac{L-1}{2}\right\}, \tag{2.3}$$

where ϵ is the lattice spacing. We denote the dimensionful extent of the torus by $\ell = \epsilon L$. One can define a discretized noncommutative torus by constructing operators Z_μ and Γ_μ, which corresponds to $\mathrm{e}^{2\pi i\hat{x}_\mu/\ell}$ and $\mathrm{e}^{\epsilon\hat{\partial}_\mu}$. These operators satisfy the commutation relations

$$Z_\mu Z_\nu = \mathrm{e}^{-2\pi i\Theta_{\mu\nu}}\,Z_\nu Z_\mu \tag{2.4}$$
$$\Gamma_\mu\,Z_\nu\,\Gamma_\mu^\dagger = \mathrm{e}^{2\pi i\epsilon\delta_{\mu\nu}/\ell}\,Z_\nu \tag{2.5}$$
$$\Gamma_\mu\Gamma_\nu = \mathcal{Z}_{\mu\nu}\Gamma_\nu\Gamma_\mu\,, \tag{2.6}$$

where $\Theta_{\mu\nu} = 2\pi\theta_{\mu\nu}/\ell^2$ is the dimensionless noncommutativity parameter and the phase factors $\mathcal{Z}_{\mu\nu} = \mathcal{Z}^*_{\nu\mu}$ are related to $c_{\mu\nu}$ as $\mathcal{Z}_{\mu\nu} = \mathrm{e}^{i\epsilon^2 c_{\mu\nu}}$.

[2]Since we will have to restrict L to be odd later, we already assume it at this point.

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In what follows we construct a representation of operators Z_μ and Γ_μ in terms of $N \times N$ unitary matrices. Let us start with the Z_μ's. The commutation (2.4) is invariant under

$$Z_\mu \mapsto \hat{g}\, Z_\mu\, \hat{g}^\dagger\,, \tag{2.7}$$

where $\hat{g}$ is an SU(N) matrix and the U(1)^D symmetry

$$Z_\mu \mapsto \mathrm{e}^{i\alpha_\mu} Z_\mu\,. \tag{2.8}$$

We assume that D is even, since we are going to introduce chiral fermions. The form of the noncommutativity matrix is taken to be $\Theta_{\mu\nu} = \frac{b}{L}\varepsilon_{\mu\nu}$, where b is an integer, and $\varepsilon_{\mu\nu}$ is a skew-diagonal antisymmetric matrix defined by

$$\varepsilon = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}. \tag{2.9}$$

We also assume that b and L are co-prime, and L divides the dimension N of the representation. Then the necessary and sufficient condition for the existence of a unique solution to (2.6) — up to the symmetries (2.7) and (2.8) — is [28]

$$N = L^{D/2}\,, \tag{2.10}$$

which we assume in what follows. An explicit solution to (2.4) is given by [28]

$$\begin{aligned} Z_{2j-1} &= \mathbb{1}_L \otimes \cdots \otimes V_L \otimes \cdots \otimes \mathbb{1}_L \\ Z_{2j} &= \mathbb{1}_L \otimes \cdots \otimes (W_L)^b \otimes \cdots \otimes \mathbb{1}_L \end{aligned} \tag{2.11}$$

for $j = 1, \ldots, D/2$. The SU(L) matrices V_L and W_L, which appear in the j-th entry of the tensor products in (2.11), are the shift and clock matrices

$$V_L = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}, \qquad W_L = \begin{pmatrix} 1 & & & & \\ & \mathrm{e}^{2\pi i/L} & & & \\ & & \mathrm{e}^{4\pi i/L} & & \\ & & & \ddots & \\ & & & & \mathrm{e}^{2\pi i(L-1)/L} \end{pmatrix} \tag{2.12}$$

obeying $V_L W_L = \mathrm{e}^{2\pi i/L}\, W_L V_L$. Since b and L are co-prime, there exists a set of integers q and s such that

$$bq + Ls = 1\,. \tag{2.13}$$

We may construct a representation of Γ_μ's as

$$\begin{aligned} \Gamma_{2j-1} &= \mathbb{1}_L \otimes \cdots \otimes (W_L)^{bq} \otimes \cdots \otimes \mathbb{1}_L \\ \Gamma_{2j} &= \mathbb{1}_L \otimes \cdots \otimes \left(V_L^\dagger\right)^q \otimes \cdots \otimes \mathbb{1}_L\,, \end{aligned} \tag{2.14}$$

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which obeys the commutation relation (2.5) and (2.6) due to (2.13). The phase factors $\mathcal{Z}_{\mu\nu}$ in (2.6) are given by

$$\mathcal{Z}_{\mu\nu} = \mathrm{e}^{2\pi i q\, \varepsilon_{\mu\nu}/L} \,. \tag{2.15}$$

2.2 Matrix-field correspondence

In this subsection we construct a map from fields to matrices associated with the introduction of space-time noncommutativity. Generally on the lattice, the noncommutativity parameter $\Theta_{\mu\nu}$ and the period matrix $\Sigma_{\mu\nu} = \ell\delta_{\mu\nu}$ should satisfy certain restrictions, which go away in the continuum limit [9, 10]. In the particular example discussed here, this amounts to requiring that $b = \pm 2$, which we assume in what follows.[3] This requires in particular that L is odd.

The plane wave $\mathrm{e}^{2\pi i m_\mu x_\mu/\ell}$, where m_μ is a D-dimensional integer vector, is mapped to $N \times N$ unitary matrices

$$J_{\vec{m}} \stackrel{\mathrm{def}}{=} \left[\prod_{\mu=1}^{D} (Z_\mu)^{m_\mu}\right] \mathrm{e}^{-\pi i \sum_{\mu<\nu} \Theta_{\mu\nu} m_\mu m_\nu} \,, \tag{2.16}$$

where the phase factor $\mathrm{e}^{-\pi i \sum_{\mu<\nu} \Theta_{\mu\nu} m_\mu m_\nu}$ is included so that

$$J_{-\vec{m}} = (J_{\vec{m}})^\dagger \,. \tag{2.17}$$

Since $(Z_\mu)^L = 1$ the matrix $J_{\vec{m}}$ is periodic with respect to m_μ with period L. Arbitrary complex-valued functions $\phi(x)$ on the periodic lattice Λ_ℓ can be mapped to a matrix by first Fourier transforming and then replacing $\mathrm{e}^{2\pi i m_\mu x_\mu/\ell}$ by $J_{\vec{m}}$. It follows that $\phi(x)$ is mapped to

$$\Phi = \frac{1}{N^2} \sum_{x \in \Lambda_\ell} \phi(x) \Delta(x) \,, \tag{2.18}$$

where the $N \times N$ matrices $\Delta(x)$ are defined as

$$\Delta(x) \stackrel{\mathrm{def}}{=} \frac{1}{N^2} \sum_{\vec{m}} J_{\vec{m}}\ \mathrm{e}^{2\pi i m_\mu x_\mu/\ell} \,. \tag{2.19}$$

Here the summation for $\vec{m}$ is taken over a Brillouin zone $-\frac{L-1}{2} \le m_\mu \le \frac{L-1}{2}$.

Note that $\Delta(x)$ is hermitian due to the property (2.17). It is periodic with respect to x_μ with period ℓ. It is easy to check that $\Delta(x)$ possesses the following

[3] Eq.(4.11) of [10], which is needed for the consistency of the noncommutative algebra of discretized coordinates, requires that b be even. The restriction explained before eq.(4.22) of [10], which is necessary for a proper discretization of the star-product, furthermore requires that $b = \pm 2$. Eq.(4.38) of [10], which is related to a property of star-gauge invariant observables, is satisfied automatically in the present case.

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properties

$$\mathrm{Tr}\,\Delta(x) = N \tag{2.20}$$

$$\sum_{x_\mu \in \Lambda_\ell} \Delta(x) = N^2 \mathbf{1}_N \tag{2.21}$$

$$\frac{1}{N}\mathrm{Tr}\left[\Delta(x)\Delta(y)\right] = N^2 \delta_{x,y}\,. \tag{2.22}$$

Due to eq. (2.22), one can invert (2.18) as

$$\phi(x) = \frac{1}{N}\,\mathrm{Tr}\left[\Phi\,\Delta(x)\right]. \tag{2.23}$$

Therefore, there is actually a one-to-one correspondence between matrices and fields in the present case. The number of degrees of freedom matches exactly due to (2.10); the matrix has N^2 elements and the corresponding field depends on L^D space-time points. Note also that using (2.20) with (2.18) or using (2.21) with (2.23), one obtains

$$\frac{1}{N}\mathrm{Tr}\,\Phi = \frac{1}{N^2}\sum_{x \in \Lambda_\ell} \phi(x)\,. \tag{2.24}$$

Summing a function $\phi(x)$ over the torus corresponds to taking the trace of the corresponding matrix Φ.

The product of two fields $\phi_1(x)$ and $\phi_2(x)$ on the noncommutative torus should be defined through the product of corresponding matrices Φ_1 and Φ_2 obtained by the map (2.18). This defines the so-called "star-product"

$$\phi_1(x) \star \phi_2(x) \stackrel{\mathrm{def}}{=} \frac{1}{N}\mathrm{Tr}\left[\Phi_1\Phi_2\Delta(x)\right]. \tag{2.25}$$

Using the definition of $\Delta(x)$, the star-product (2.25) can be written explicitly in terms of $\phi_\alpha(x)$ as

$$\phi_1(x) \star \phi_2(x) = \frac{1}{N^2}\sum_{y \in \Lambda_\ell}\sum_{z \in \Lambda_\ell} \phi_1(y)\phi_2(z)\mathrm{e}^{-2i(\theta^{-1})_{\mu\nu}(x_\mu - y_\mu)(x_\nu - z_\nu)}\,, \tag{2.26}$$

where $\theta_{\mu\nu}$ is the dimensionful noncommutativity parameter given by

$$\theta_{\mu\nu} = \frac{1}{2\pi}\ell^2\Theta_{\mu\nu} = -\frac{1}{2\pi}bL\epsilon^2\varepsilon_{\mu\nu}\,. \tag{2.27}$$

In the continuum, the star-product is usually written as

$$\phi_1(x) \star \phi_2(x) = \phi_1(x)\exp\left(i\frac{1}{2}\theta_{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}\right)\phi_2(x)\;, \tag{2.28}$$

which can be rewritten in an integral form as

$$\phi_1(x) \star \phi_2(x) = \frac{1}{\pi^D|\det\theta|}\iint \mathrm{d}^D y\ \mathrm{d}^D z\ \phi_1(y)\,\phi_2(z)\ \mathrm{e}^{-2i(\theta^{-1})_{\mu\nu}(x-y)_\mu(x-z)_\nu}\,. \tag{2.29}$$

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One can easily check that the star-product (2.26) is a proper discretized version of (2.29). The discretized star-product (2.26) enjoys all the algebraic properties of the continuum star-product (2.28). In particular,

$$\sum_{x_i \in \Lambda_\ell} \phi_1(x) \star \phi_2(x) \star \cdots \star \phi_n(x) \tag{2.30}$$

is invariant under cyclic permutations of $\phi_\alpha(x)$ and

$$\sum_{x_i \in \Lambda_\ell} \phi_1(x) \star \phi_2(x) = \sum_{x_i \in \Lambda_\ell} \phi_1(x)\phi_2(x)\,. \tag{2.31}$$

Finally let us mention the properties of the shift operator Γ_μ, which is expressed in terms of matrices as (2.14). From (2.5) it follows that

$$\Gamma_\mu \Delta(x) \Gamma_\mu^\dagger = \Delta(x - \epsilon\hat{\mu})\,, \tag{2.32}$$

which implies

$$\phi(x + \epsilon\hat{\mu}) = \frac{1}{N} \operatorname{Tr}\left[\Gamma_\mu \Phi \Gamma_\mu^\dagger \Delta(x)\right]. \tag{2.33}$$

Let us consider the field $S_\mu(x)$, which corresponds to the matrix Γ_μ.

$$S_\mu(x) = \frac{1}{N} \operatorname{Tr}\left[\Gamma_\mu\, \Delta(x)\right] = \exp\left(i2\pi \frac{L-1}{2} \varepsilon_{\mu\nu} \frac{x_\nu}{\ell}\right). \tag{2.34}$$

Such a field has the property

$$S_\mu(x) \star \phi(x) \star S_\mu(x)^* = \phi(x + \epsilon\hat{\mu})\,, \tag{2.35}$$

for an arbitrary field $\phi(x)$. Obviously this property is quite peculiar to noncommutative geometry. It plays an important role in the construction of chiral gauge theories as we shall see in section 3.

2.3 Noncommutative Yang-Mills theories on the lattice

Let us define noncommutative U(r) Yang-Mills theory on the lattice. As in commutative lattice gauge theories, we introduce the link variables $U_\mu(x)$, which can be regarded as $r \times r$ matrix fields on the periodic lattice Λ_ℓ. The unitarity condition of the link variables should be naturally replaced by

$$U_\mu(x)^\dagger \star U_\mu(x) = U_\mu(x) \star U_\mu(x)^\dagger = \mathbb{1}_r \qquad \text{(no summation over } \mu)\,, \tag{2.36}$$

which means that the matrix field $U_\mu(x)$ is 'star-unitary'. Note that the matrix $U_\mu(x)$ for a given x is not necessarily unitary. Let us introduce $n \times n$ matrices $\hat{U}_\mu$, where $n = rN$, by

$$\hat{U}_\mu = \frac{1}{N^2} \sum_{x \in \Lambda_\ell} U_\mu(x) \otimes \Delta(x)\,. \tag{2.37}$$

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Due to (2.36), the matrix $\hat{U}_\mu$ should be unitary. One can invert (2.37) by using (2.22) as

$$U_\mu(x) = \frac{1}{N}\,\mathrm{Tr}\left[\hat{U}\,\Delta(x)\right], \tag{2.38}$$

where we note that the trace Tr is taken over N-dimensional indices corresponding to the 'space-time coordinates' only. Therefore, there is a one-to-one correspondence between $r \times r$ star-unitary matrix field $U_\mu(x)$ and $n \times n$ unitary matrix $\hat{U}$.

One can write down the action for the noncommutative Yang-Mills theory as

$$S_{\mathrm{NCYM}} = -\beta\epsilon^D \sum_{x\in\Lambda_\ell}\sum_{\mu\neq\nu} \mathrm{tr}\left[U_\mu(x)\star U_\nu(x+\epsilon\hat{\mu})\star U_\mu(x+\epsilon\hat{\nu})^\dagger \star U_\nu(x)^\dagger\right], \tag{2.39}$$

where the trace tr is taken over the gauge indices. The action (2.39) is invariant under the star-gauge transformation

$$U_\mu(x) \mapsto g(x)\star U_\mu(x)\star g(x+\epsilon\hat{\mu})^\dagger, \tag{2.40}$$

where the gauge function $g(x)$ is star-unitary, $g(x)\star g(x)^\dagger = g(x)^\dagger \star g(x) = \mathbb{1}_r$.

Let us rewrite the action (2.39) in terms of the matrices. Defining the unitary matrices $\hat{V}_\mu$ as

$$\hat{V}_\mu = \hat{U}_\mu \Gamma_\mu\,, \tag{2.41}$$

where Γ_μ are the SU(N) matrices defined by (2.14), we arrive at

$$S_{\mathrm{NCYM}} = -N\beta\sum_{\mu\neq\nu}\mathcal{Z}_{\mu\nu}\,\mathrm{Tr}\,\mathrm{tr}\left(\hat{V}_\mu\,\hat{V}_\nu\,\hat{V}_\mu^\dagger\,\hat{V}_\nu^\dagger\right), \tag{2.42}$$

where the phase factor $\mathcal{Z}_{\mu\nu}$ is given by (2.15).[4] The action (2.42) is nothing but the twisted Eguchi-Kawai model [26, 27]. The star-gauge invariance (2.40) of (2.39) corresponds to the SU(n) invariance

$$\hat{V}_\mu \mapsto \hat{g}\,\hat{V}_\mu\,\hat{g}^\dagger\,, \tag{2.43}$$

of the action (2.42), where $\hat{g}$ is an SU(n) matrix.

The integration measure for the U(n) matrices $\hat{U}_\mu$ of the twisted Eguchi-Kawai model is given by the Haar measure of the U(n) group, which respects the SU(n) invariance (2.43). This naturally defines a star-gauge invariant measure for the star-unitary matrix field $U_\mu(x)$. Thus the twisted Eguchi-Kawai model can be interpreted as a noncommutative U(r) Yang-Mills theory on the periodic lattice.

Finally, we comment on the continuum limit $\epsilon \to 0$. We recall that the dimensionful noncommutativity parameter $\theta_{\mu\nu}$ is given by (2.27). Therefore, in order to obtain a finite value of $\theta_{\mu\nu}$, we have to take the large $N(=L^{D/2})$ limit simultaneously

[4] Throughout the paper, we use Tr for the trace over N-dimensional indices corresponding to the 'space-time coordinates', and tr for the trace over r-dimensional gauge indices.

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with the continuum limit $\epsilon \to 0$ fixing $L\epsilon^2$. Note that such a large N limit is different from the one needed to reproduce ordinary large N gauge theories from twisted Eguchi-Kawai models [27], where one takes the large N limit first for fixed ϵ followed by the continuum limit $\epsilon \to 0$. (In either large N limit, the coupling constant β in (2.42) should be tuned properly as a function of ϵ.) Existence of the new large N limit with fixed $L\epsilon^2$ has been observed by Monte Carlo simulation in $D = 2$ [29]. In this limit, the size of the torus $\ell = \epsilon L$ goes to infinity. In order to have both $\theta_{\mu\nu}$ and ℓ finite, one has to use a more general construction [8, 9, 10].

3. Chiral fermions on the noncommutative torus

In this section, we incorporate chiral fermions in the lattice noncommutative gauge theory described in the previous section. If one introduces fermions naively, one encounters the doubling problem as in ordinary lattice gauge theories.[5] This makes the implementation of chiral fermions nontrivial.

In ref. [33], the Eguchi-Kawai model was used to define a unitary matrix version of the IIB matrix model, which describes a toroidal compactification of the target space of type IIB superstrings. Chiral fermion without species doublers has been implemented with the formalism of ref. [35], which is actually equivalent to the more recent approach [14], except for the choice of the fermion measure. In particular, the implementation maintains the $\mathrm{SU}(n)$ invariance (2.43). Recalling that the star-gauge invariance of noncommutative gauge theories corresponds to the $\mathrm{SU}(n)$ invariance (2.43) of the twisted Eguchi-Kawai model, it is suggested that we can implement chiral fermions in noncommutative gauge theories with manifest star-gauge invariance. We will see this explicitly in what follows.

3.1 Lattice Dirac fermions with manifest chiral symmetry

Let us introduce Dirac fields $\psi(x)$, $\bar{\psi}(x)$ on the periodic lattice Λ_ℓ. In the case of the fundamental representation, the fields $\psi(x)$ and $\bar{\psi}(x)$ are r-component column and row vectors, respectively, in the internal space, and transform under star-gauge transformation (2.40) as

$$\psi(x) \mapsto g(x) \star \psi(x) \, ; \qquad \bar{\psi}(x) \mapsto \bar{\psi}(x) \star g(x)^\dagger \, . \tag{3.1}$$

We introduce the forward and backward covariant derivative on the noncommutative torus as

$$\begin{aligned} \nabla_\mu \psi &= \frac{1}{\epsilon}\Big[U_\mu(x) \star \psi(x + \epsilon\hat{\mu}) - \psi(x)\Big] \\ \nabla^*_\mu \psi &= \frac{1}{\epsilon}\Big[\psi(x) - U_\mu(x - \epsilon\hat{\mu})^\dagger \star \psi(x - \epsilon\hat{\mu})\Big] \, . \end{aligned} \tag{3.2}$$

[5] The fermion doubling problem is addressed in refs. [30, 31, 32] in the context of noncommutative geometry, and in refs. [33, 34] in matrix models.

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One can define an action for Dirac fermion as

$$S_{\rm w} = \epsilon^D \sum_x \bar{\psi}(x) \, \star D_{\rm w} \, \psi(x) \,, \tag{3.3}$$

where $D_{\rm w}$ is the Wilson-Dirac operator given as

$$D_{\rm w} = \frac{1}{2} \sum_{\mu=1}^{D} \{ \gamma_\mu (\nabla^*_\mu + \nabla_\mu) + \epsilon \nabla^*_\mu \nabla_\mu \} \,. \tag{3.4}$$

The star ($\star$) written explicitly in (3.3) — but not the ones hidden in the operator $D_{\rm w}$ — can be omitted due to the property (2.31). For trivial gauge configuration $U_\mu(x) = \mathbb{1}_r$, the action (3.3) agrees with the usual Wilson-Dirac action in the commutative space-time. The $\mathrm{O}(\epsilon)$ term in (3.4) is the Wilson term, which is introduced to give the unwanted species doublers masses of $\mathrm{O}(\epsilon^{-1})$. However, as is well known, the Wilson term breaks chiral symmetry.

In the case of the adjoint representation, the Dirac fields are $r \times r$ matrices in the internal space, and transform under star-gauge transformation (2.40) as

$$\psi(x) \mapsto g(x) \star \psi(x) \star g(x)^\dagger \,; \qquad \bar{\psi}(x) \mapsto g(x) \star \bar{\psi}(x) \star g(x)^\dagger \,. \tag{3.5}$$

The forward and backward covariant derivative can be defined as

$$\begin{aligned} \nabla_\mu \psi &= \frac{1}{\epsilon} \Big[U_\mu(x) \star \psi(x + \epsilon\hat{\mu}) \star U_\mu(x)^\dagger - \psi(x) \Big] \\ \nabla^*_\mu \psi &= \frac{1}{\epsilon} \Big[\psi(x) - U_\mu(x - \epsilon\hat{\mu})^\dagger \star \psi(x - \epsilon\hat{\mu}) \star U_\mu(x - \epsilon\hat{\mu}) \Big] \,. \end{aligned} \tag{3.6}$$

One can define an action for Dirac fermion as

$$S_{\rm w} = \epsilon^D \sum_x \mathrm{tr} \left[\bar{\psi}(x) \, \star D_{\rm w} \, \psi(x) \right] . \tag{3.7}$$

As in the commutative case [15], we can maintain an exact chiral symmetry on the lattice without doublers by using a Dirac operator which satisfies the Ginsparg-Wilson relation.

$$\gamma_5 D + D \gamma_5 = \epsilon \, D \gamma_5 D \,. \tag{3.8}$$

Assuming $D^\dagger = \gamma_5 D \gamma_5$, which is usually refered to as "γ_5-Hermiticity", we can define the unitary operator $\hat{\gamma}_5$

$$\hat{\gamma}_5 = \gamma_5 (1 - \epsilon D) \,, \tag{3.9}$$

which has the properties

$$(\hat{\gamma}_5)^2 = 1 \tag{3.10}$$

$$\gamma_5 D = - D \hat{\gamma}_5 \,. \tag{3.11}$$

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Eq. (3.11) implies that the corresponding action

$$S = \epsilon^D \sum_x \bar{\psi}(x) \star D\psi(x) \qquad \text{for fundamental representation} \tag{3.12}$$

$$S = \epsilon^D \sum_x \mathrm{tr}\left[\bar{\psi}(x) \star D\psi(x)\right] \qquad \text{for adjoint representation} \tag{3.13}$$

has the symmetry

$$\psi(x) \mapsto \mathrm{e}^{i\alpha\hat{\gamma}_5}\psi(x)\,; \qquad \bar{\psi}(x) \mapsto \bar{\psi}(x)\mathrm{e}^{i\alpha\gamma_5}\,, \tag{3.14}$$

which is the exact lattice chiral symmetry.

As in the commutative case [16], an explicit solution to the Ginsparg-Wilson relation (3.8), which transforms covariantly under star-gauge transformation, can be given by

$$D = \frac{1}{\epsilon}\left\{1 - A(A^\dagger A)^{-\frac{1}{2}}\right\}, \qquad A = 1 - \epsilon D_{\mathrm{w}}\,, \tag{3.15}$$

where the Wilson-Dirac operator D_{w} is defined by eq. (3.4).

3.2 Projecting out chiral fermions

Due to the property (3.11), we can project a chiral fermion out of Dirac fermion as in [14] by imposing the constraint

$$\hat{\gamma}_5\,\psi(x) = \psi(x) \tag{3.16}$$

$$\bar{\psi}(x)\,\gamma_5 = -\bar{\psi}(x)\,. \tag{3.17}$$

In order to define the integration measure for $\psi(x)$ and $\bar{\psi}(x)$ after the projection (3.16) and (3.17), we take a complete orthogonal basis $\{\varphi_j(x); j = 1, \cdots, K\}$ and $\{\bar{\varphi}_j(x); j = 1, \cdots, \bar{K}\}$ for the solutions to (3.16) and (3.17) respectively as

$$\hat{\gamma}_5\,\varphi_j(x) = \varphi_j(x)\;; \qquad (\varphi_j, \varphi_k) \stackrel{\text{def}}{=} \sum_{x\in\Lambda_\ell} \varphi_j(x)^\dagger\,\varphi_k(x) = \delta_{jk} \tag{3.18}$$

$$\bar{\varphi}_j(x)\,\gamma_5 = -\bar{\varphi}_j(x)\;; \qquad (\bar{\varphi}_j, \bar{\varphi}_k) = \delta_{jk}\,. \tag{3.19}$$

The general solution to (3.16) and (3.17) can be written in terms of the complete basis as

$$\psi(x) = \sum_j c_j\,\varphi_j(x)\,; \qquad \bar{\psi}(x) = \sum_j \bar{c}_j\,\bar{\varphi}_j(x)\,, \tag{3.20}$$

where the coefficients c_j and $\bar{c}_j$ are Grassmann variables. Then the integration measure for the chiral fermion fields can be defined by

$$\mathcal{D}\psi\,\mathcal{D}\bar{\psi} = \prod_j \mathrm{d}c_j \cdot \prod_k \mathrm{d}\bar{c}_k\,. \tag{3.21}$$

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Here we recall that unlike the usual chirality operator, $\hat{\gamma}_5$ depends on the gauge configuration $U_\mu(x)$. Therefore, the basis $\{\varphi_j(x)\}$ has to be specified for each gauge configuration $U_\mu(x)$, whereas one can take the same basis $\{\bar{\varphi}_j(x)\}$ for all gauge configurations. If we choose a different basis $\{\varphi_j(x)\}$, the integration measure defined by (3.21) may change by a phase factor which depends on the gauge configuration. Therefore, the crucial question that arises here is whether one can fix this gauge-field dependent phase ambiguity in such a way that the integration measure is star-gauge invariant.

We fix the gauge-field dependent phase ambiguity in the following way. First we specify the complete basis for the gauge configuration $U_\mu^{(0)}(x)$ as $\{\varphi_j^{(0)}(x)\}$. Then for a general gauge configuration $U_\mu(x)$, we require that the basis $\{\varphi_j(x)\}$ satisfy the condition

$$\det_{j,k}\left\{(\varphi_j^{(0)}, \varphi_k)\right\} \ : \text{real positive} \, . \tag{3.22}$$

Although this does not determine the basis $\{\varphi_j(x)\}$ uniquely, it determines the integration measure uniquely (up to a constant phase factor) as far as the determinant is non-zero. This phase choice was originally proposed in ref. [35] for commutative lattice gauge theory, where the 'reference configuration' $U_\mu^{(0)}(x)$ was taken to be $U_\mu^{(0)}(x) = \mathbb{1}_r$. However, the corresponding fermion measure is *not* gauge invariant, due to the fact that the configuration $U_\mu^{(0)}(x) = \mathbb{1}_r$ is not invariant under the gauge transformation.[6]

In the noncommutative space-time, there exists a gauge configuration $\tilde{U}_\mu(x)$ which is invariant under arbitrary star-gauge transformations.

$$g(x) \star \tilde{U}_\mu(x) \star g(x + \epsilon\hat{\mu})^\dagger = \tilde{U}_\mu(x) \, . \tag{3.23}$$

The unique solution to (3.23) — up to a constant phase — is $\tilde{U}_\mu(x) = S_\mu(x)^* \mathbb{1}_r$, where $S_\mu(x)$ is the field defined in (2.34), and the invariance (3.23) follows from the property (2.35). We are going to use this configuration $\tilde{U}_\mu(x)$ as the reference configuration; namely $U_\mu^{(0)}(x) = \tilde{U}_\mu(x)$.

Let us recall that the dimension K of the solution space of (3.16) depends on the gauge configuration. Here we focus on configurations U for which $K[U] = \bar{K}$, namely we restrict ourselves to the topologically trivial sector of gauge configurations (i.e., there is no baryon number violation). In order for the condition (3.22) to work, we should have $K[U] = K[U^{(0)}]$, which requires $K[U^{(0)}] = \bar{K}$. In fact, this is guaranteed for the adjoint representation, but not for the fundamental representation. Let us consider the adjoint representation first. For the particular configuration $U_\mu(x) =$

[6]In ref. [35], it was speculated that the gauge averaging will lead to a sensible definition of a chiral gauge theory when the fermion content satisfies the anomaly cancellation condition. The two-dimensional U(1) case, where analytic results are available, has been studied intensively in [36], where a particular class of gauge anomaly, which may contribute significantly in the gauge averaging, has been identified. See also ref. [37] for a clarification on this point.

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$U^{(0)}_\mu(x)$, one finds that the covariant derivatives ∇_μ and ∇^*_μ vanish and hence we obtain[7] $\hat{\gamma}_5 = \gamma_5$, which means in particular that $K[U^{(0)}] = \bar{K}$ as announced. Since the determinant in (3.22) is nonzero for generic configurations in the topologically trivial sector, the condition (3.22) works in the adjoint representation. In the case of the fundamental representation, however, the covariant derivatives ∇_μ and ∇^*_μ do not vanish for the configuration $U_\mu(x) = U^{(0)}_\mu(x)$ and there is no reason for $K[U^{(0)}] = \bar{K}$. In fact we have checked numerically that $K[U^{(0)}] \neq \bar{K}$ in general. Therefore, the condition (3.22) does not work in the fundamental representation.

3.3 Star-gauge invariance of the fermion measure

For the reason mentioned in the previous subsection, we restrict ourselves to the adjoint representation in what follows. Let us prove that the corresponding integration measure is indeed star-gauge invariant. Here we denote the basis as $\varphi_j[U]$, to express the gauge field dependence manifestly and suppress the space-time argument x. Then

$$\begin{aligned} g \star \varphi_j[U] \star g^\dagger &= \sum_k \varphi_k[U^g]\, Q_{kj}[U,g] \\ g \star \varphi^{(0)}_j \star g^\dagger &= \sum_k \varphi^{(0)}_k\, Q^{(0)}_{kj}[g]\,, \end{aligned} \tag{3.24}$$

where U^g represents the star-gauge transformed configuration and Q_{kj} and $Q^{(0)}_{kj}$ are *unitary* transformation matrices. It follows that

$$\begin{aligned} \det_{j,k}\{(\varphi^{(0)}_j, \varphi_k(U))\} &= \det_{j,k}\Big\{(g \star \varphi^{(0)}_j \star g^\dagger, g \star \varphi_k[U] \star g^\dagger)\Big\} \\ &= \det_{j,k}\Big\{\sum_{m,n} (Q^{(0)}_{mj})^* (\varphi^{(0)}_m, \varphi_n[U^g]) Q_{nk}\Big\} \\ &= (\det Q^{(0)})^* \cdot \det_{j,k}\Big\{(\varphi^{(0)}_j, \varphi_k[U^g])\Big\} \cdot \det Q\,. \end{aligned} \tag{3.25}$$

Due to the condition (3.22), we obtain $\det Q = \det Q^{(0)}$. Writing the transformed fermion field $\psi' = g \star \psi \star g^\dagger$ in two different ways

$$\begin{aligned} \psi' &= \sum_j c'_j\, \varphi_j[U^g] \\ \psi' &= \sum_j c_j\, g \star \varphi_j[U] \star g^\dagger = \sum_{j,k} c_j\, \varphi_k[U^g]\, Q_{kj}\,, \end{aligned} \tag{3.26}$$

[7] From this, it also follows that we can use the usual chiral basis as $\{\varphi^{(0)}_j(x)\}$ in (3.22). In fact, the basis $\{\varphi^{(0)}_j(x)\}$ is all we need to specify the fermion measure. The particular configuration $\tilde{U}_\mu(x)$, which is not a smooth function of x (See eq. (2.34)), is used for convenience in the proof of the star-gauge invariance, but it does not play any role in specifying the measure.

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we obtain the transformation of the coefficients c_j as

$$c'_j = \sum_k Q_{jk}\, c_k\,. \tag{3.27}$$

Therefore, the measure for ψ fields transforms as

$$\mathcal{D}\psi' = \prod_j \mathrm{d}c'_j = (\det Q[U,g])^{-1} \prod_j \mathrm{d}c_j = (\det Q^{(0)}[g])^{-1}\, \mathcal{D}\psi\ . \tag{3.28}$$

Similarly, the measure for $\bar{\psi}$ fields transforms as

$$\mathcal{D}\bar{\psi}' = (\det \bar{Q}[g])^{-1}\, \mathcal{D}\bar{\psi}\,, \tag{3.29}$$

where the unitary matrix $\bar{Q}_{kj}[g]$ is defined by

$$g \star \bar{\varphi}_j \star g^\dagger = \sum_k \bar{\varphi}_k\, \bar{Q}_{kj}[g]\,. \tag{3.30}$$

It is easy to check that $\det Q^{(0)}[g] = (\det \bar{Q}[g])^*$ for arbitrary g, from which it follows that the fermion measure (3.21) is star-gauge invariant.

3.4 Chiral determinants

Let us consider the fermionic partition function defined as

$$Z[U] = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathrm{e}^{-S}\,, \tag{3.31}$$

where the action and the measure is given by (3.13) and (3.21) respectively. One can write it explicitly as

$$Z[U] = \det_{j,k}\Big\{(\bar{\varphi}_j, \varphi_k[U])\Big\}\,, \tag{3.32}$$

with the condition (3.22). This quantity actually vanishes identically due to the fact that the constant modes $\psi(x) \propto \mathbb{1}_r$ and $\bar{\psi}(x) \propto \mathbb{1}_r$ do not appear in the action. (Recall that we are considering the adjoint representation.)

In order to define a nonzero partition function we have to insert these modes as external lines as

$$Z'[U] = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi} \prod_{\alpha=p/2+1}^{p} \Big\{\sum_x \mathrm{tr}\,\Big[\bar{\psi}_\alpha(x)\Big]\Big\} \prod_{\beta=1}^{p/2}\Big\{\sum_y \mathrm{tr}\,\Big[\psi_\beta(y)\Big]\Big\}\, \mathrm{e}^{-S}\,, \tag{3.33}$$

where the integer p is the dimension of the (Dirac) spinor space $p = 2^{D/2}$ and we took the Weyl representation $\gamma_5 = \mathrm{diag}(1,\cdots,1,-1,\cdots,-1)$. The partition function $Z'[U]$ can be written in the same form as (3.32) with the same condition (3.22), but the bases in (3.32) and (3.22) should now be constructed in the subspace orthogonal to the direction of the constant modes $\psi(x) \propto \mathbb{1}_r$ and $\bar{\psi}(x) \propto \mathbb{1}_r$. The partition

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function $Z'[U]$ is nonzero for generic configurations, and moreover it is invariant under the star-gauge transformation:

$$Z'[U^g] = Z'[U]\,. \tag{3.34}$$

The fermion determinant $Z'[U]$ can be calculated numerically for each gauge configuration U in the following way. In order to obtain the basis $\{\varphi_j[U]\}$, it is convenient to note that

$$\hat{\gamma}_5 = \gamma_5 A (A^\dagger A)^{-\frac{1}{2}} = \frac{H}{\sqrt{H^\dagger H}}\,, \tag{3.35}$$

where H is a hermitian operator defined as

$$H = \gamma_5 A = \gamma_5 (1 - \epsilon D_{\mathrm{w}})\,. \tag{3.36}$$

Then instead of using (3.18), one can construct the basis $\{\varphi_j[U]\}$ as the eigenvectors of the hermitian operator H

$$H\varphi_j = E_j \varphi_j \tag{3.37}$$

with positive eigenvalues $E_j > 0$ [35]. Actual calculations become very simple by rewriting equations such as (3.37) using the matrix-field correspondence described in section 2.2. We calculate the fermion determinant $Z'[U]$ in the $D = 2$, U(1) case. The parameters in (2.13) are chosen to be $b = 2$, $s = -1$ and $q = \frac{L+1}{2}$. For $L = 3, 5, 7, 9, 11$, we have checked that $Z'[U]$ has a nontrivial phase, which is invariant under random star-gauge transformations.

4. Gauge anomalies in the continuum calculations

In this section, we will review some relevant aspects of gauge anomalies in noncommutative chiral gauge theories and compare our results on the lattice to some continuum calculations. Gauge anomalies in noncommutative chiral gauge theories has been extensively studied in the literature recently, mostly restricted to four dimensions [20, 23, 24, 38].

One of the intriguing features of anomalies in noncommutative chiral gauge theories is the fact that the anomaly cancellation condition differs from the one encountered in the commutative version of the theory, yet it is independent of the noncommutativity parameter. This can be seen as a consequence of UV/IR mixing [39] reflecting the noncommutativity between the two limits $\theta_{\mu\nu} \to 0$ and $\Lambda_{\mathrm{UV}} \to \infty$. In computing the anomalous Ward identities for the gauge current in noncommutative chiral gauge theories one recovers the ordinary commutative anomaly by taking the commutative limit while keeping the ultraviolet cutoff fixed; once the the ultraviolet cutoff is removed the limit $\theta_{\mu\nu} \to 0$ does not retrieve the commutative Ward identity anymore.

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4.1 Generalities

We consider noncommutative $\mathrm{U}(r)$ gauge theory in the continuum. The gauge field $A_\mu(x)$ is a $r \times r$ hermitian matrix field, which transform as

$$\delta_\eta A_\mu = \partial_\mu \eta - i(A_\mu \star \eta - \eta \star A_\mu)\,, \tag{4.1}$$

where the $r \times r$ hermitian matrix field $\eta(x)$ represents the gauge function. For the fermion fields ψ, $\bar{\psi}$ transforming in the (anti) fundamental representation of $\mathrm{U}(r)$ as

$$\begin{array}{ll} \delta_\eta \psi = i\eta \star \psi\,; & \delta_\eta \bar{\psi} = -i\bar{\psi} \star \eta \\ \delta_\eta \psi = -i\psi \star \eta\,; & \delta_\eta \bar{\psi} = i\eta \star \bar{\psi}\,, \end{array} \tag{4.2}$$

the covariant derivatives are given respectively by

$$\begin{aligned} D_\mu \psi &= \partial_\mu \psi - iA_\mu \star \psi\ , \\ D_\mu \psi &= \partial_\mu \psi + i\psi \star A_\mu\ \ . \end{aligned} \tag{4.3}$$

In the $\mathrm{U}(1)$ case, (anti) fundamental fermions have respectively charge ± 1 with respect to the gauge field; any other charge assignment would spoil the covariance of (4.3) under (4.2). For the fermion fields transforming in the adjoint representation as

$$\delta_\eta \psi = i(\eta \star \psi - \psi \star \eta)\,; \qquad \delta_\eta \bar{\psi} = i(\eta \star \bar{\psi} - \bar{\psi} \star \eta)\,, \tag{4.4}$$

the covariant derivative is given by

$$D_\mu \psi = \partial_\mu \psi - i(A_\mu \star \psi - \psi \star A_\mu)\,. \tag{4.5}$$

4.2 Review of the four-dimensional case

When $D = 4$ the calculation of the gauge anomaly in noncommutative chiral gauge theories with fermions in the (anti) fundamental representation has been done using either the heat-kernel method [20] or the perturbative analysis [24], leading to the following anomaly cancellation condition

$$\mathrm{tr}\,\left(T^a T^b T^c\right) = 0\,, \tag{4.6}$$

where T^a are the generators of the representation of the gauge group. Notice that this condition is much stronger that the one for commutative chiral gauge theories, $\mathrm{tr}\,\left(T^a\{T^b, T^c\}\right) = 0$. One particular feature of noncommutative chiral gauge theories with the (anti) fundamental coupling is that, unlike the case of their commutative counterparts, the form of eq. (4.6) implies that the only way to cancel the anomaly by adding several types of fermions is to have the same number of fermions of one chirality transforming in the fundamental and antifundamental representations (or equivalently the same number of left- and right-handed fermions with the same transformation properties). This is easy to see when the gauge group is $\mathrm{U}(1)$ [40] and

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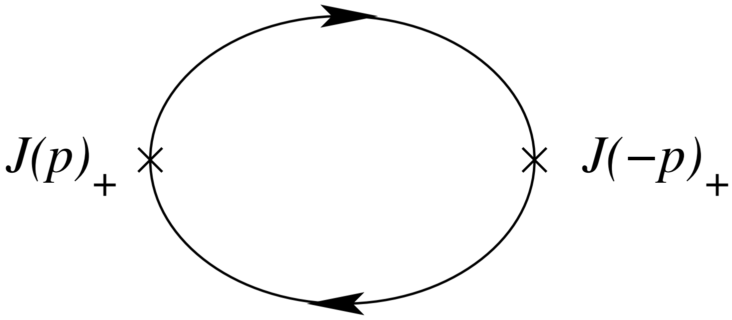

Figure 1: Feynman diagram contributing to the two-dimensional gauge anomaly.

the anomaly cancellation condition reduces to the one for commutative chiral QED, $\sum_i q_i^3 = 0$. Now, however, since the charges are fixed by the transformation properties of the fermion the only way to cancel the total anomaly is by considering pairs of fermions transforming respectively in the fundamental and the antifundamental representation (or again, pairs of fermions with opposite chirality and the same U(1) charge). The resulting theory is therefore vector-like.

The situation changes when the fermions transform in the adjoint representation of U(r). A direct calculation of the triangle anomaly shows that noncommutative chiral gauge theories with adjoint fermions are anomaly free [24, 38]. This is not so surprising, though, since four-dimensional chiral fermions in the adjoint representation can be alternatively formulated as Majorana fermions. Our result in the previous section using the lattice construction of the noncommutative chiral gauge theory is consistent with these observations.

4.3 The two-dimensional case

The analysis of section 3 is valid in any even dimension and therefore noncommutative chiral gauge theories with adjoint fermions can be constructed on the lattice without breaking of star-gauge invariance for any $D = 2k$. Thus it would be interesting to see whether the cancellation of anomalies for adjoint fermions in the continuum found in four dimensions also happens in other dimensions different from four. The simplest case is a gauge theory with adjoint chiral fermions in Euclidean two-dimensional noncommutative space-time. In two dimensions the analog of the triangle diagram is the chiral fermion loop with two gauge field insertions (see figure 1). Because we are in two dimensions the chirality matrix γ_5 satisfies the identity $\gamma_\mu\gamma_5 = i\epsilon_{\mu\nu}\gamma_\nu$. Therefore we can write[8]

$$\gamma_\mu \mathrm{P}_\pm = \frac{1}{2}(\delta_{\mu\nu} \pm i\epsilon_{\mu\nu})\gamma_\nu\,, \tag{4.7}$$

where $\mathrm{P}_\pm = \frac{1}{2}(1 \pm \gamma_5)$ are the projectors over right(left)-handed chiralities. As a consequence the amplitude with chiral couplings to the gauge field in figure 1,

[8]We use the conventions $\gamma_5 = i\gamma_0\gamma_1$, $\gamma_\mu^\dagger = \gamma_\mu$.

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$\Gamma^{ab}_{\mu\nu}(p^2)_+$, is related to the contribution of the same diagram with vector-like couplings $\Gamma^{ab}_{\mu\nu}(p^2)$ by

$$\Gamma^{ab}_{\mu\nu}(p^2)_+ = \frac{1}{2}\left(\delta_{\mu\alpha} + i\epsilon_{\mu\alpha}\right)\Gamma^{ab}_{\alpha\nu}(p^2)\,. \tag{4.8}$$

The term $\Gamma^{ab}_{\mu\nu}(p^2)$ has been computed in [41, 42] where it was seen that it only receives contributions from the planar part of the amplitude. Using that result we find that the chiral anomaly is given by

$$p_\mu\Gamma^{ab}_{\mu\nu}(p^2)_+ = \frac{i}{4\pi}(d^{ace}d^{bce} + f^{ace}f^{bce})\epsilon_{\mu\nu}p_\mu\,. \tag{4.9}$$

Here $f^{abc} = -2i\,\mathrm{tr}\,(T^a[T^b, T^c])$ are the structure constants and $d^{abc} = 2\,\mathrm{tr}\,(T^a\{T^b, T^c\})$ are the anomaly coefficients of the gauge group (with the normalization $\mathrm{tr}\,(T^aT^b) = \frac{1}{2}\delta^{ab}$).

For $\mathrm{U}(r)$ a simple calculation leads to $d^{ace}d^{bce} + f^{ace}f^{bce} = 2r\delta^{ab}$ and we find the following anomalous Ward identity

$$p_\mu\langle J^a_\mu(p)_+\, J^b_\nu(-p)_+\rangle = \frac{ir}{2\pi}p_\mu\epsilon_{\mu\nu}\delta^{ab}\,, \tag{4.10}$$

where $J^a_\mu(p)_+$ is the chiral gauge current derived from the lagrangian $\mathcal{L} = \mathrm{tr}\,\left(\bar\psi\star\gamma_\mu D_\mu\,\mathrm{P}_+\psi\right)$ with the covariant derivative defined in (4.5). Notice that, as in the four-dimensional case, even if the anomalous Ward identity (4.9) is independent of the noncommutativity parameter the actual form of the anomaly differs from the one corresponding to a commutative chiral gauge theory, where the term $d^{ace}d^{bce}$ would be absent from the group theoretical factor. Again this is a consequence of UV/IR mixing. However, when the gauge group is $\mathrm{U}(r)$ with $r > 1$, the actual numerical value of the prefactor on the right-hand side of (4.10) coincides with the one for the corresponding commutative gauge theory. This is due to the $\mathrm{U}(r)$ identity $f^{ace}f^{bce} = d^{ace}d^{bce} = r\delta^{ab}$. Thus, the contribution from the d-symbols compensates the factor $\frac{1}{2}$ that comes from the fact that only the planar part contributes to the diagram in figure 1.[9]

In the case when the gauge group is $\mathrm{U}(1)$, in order to connect with the literature [43], it is convenient to change the normalization of the only generator from $T^0 = \frac{1}{\sqrt{2}}$ to $T^0 = 1$. Thus, the anomaly can be obtained from eq. (4.9) by setting $f^{000} = 0$ and $d^{000} = 2$. The interesting thing about this case is that the naive $\theta_{\mu\nu} \to 0$ limit in the classical action gives a free theory, whereas the calculation of the anomaly in the noncommutative case gives a θ-independent nonvanishing result which resembles that of a two-dimensional nonabelian commutative gauge theory (cf. [43, 42]).

[9] Alternatively, this can be seen by noticing that for $\mathrm{U}(r)$ $(r > 1)$ all dependence on θ in the calculation of the anomaly disappears from the very beginning since $d^{ace}d^{bce}\sin^2\frac{1}{2}\theta(p,q) + f^{ace}f^{bce}\cos^2\frac{1}{2}\theta(p,q) = r\delta^{ab}$.

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We have seen that in the case of two-dimensional noncommutative chiral gauge theories with fermions in the adjoint representation there is a gauge anomaly. One way to cancel this anomaly is by adding a second adjoint fermion with opposite chirality, with Dirac operator $\gamma_\mu D_\mu \mathrm{P}_-$. As a consequence, the resulting theory will be nonchiral and will satisfy the vector Ward identity $p_\mu \langle J^a_\mu(p) J^b_\nu(-p)\rangle \equiv p_\mu \Gamma^{ab}_{\mu\nu}(p^2) = 0$, with $J^a_\mu(p) = J^a_\mu(p)_+ + J^a_\mu(p)_-$. This perturbative result in the continuum contrasts with our construction on the lattice where gauge invariance is preserved without renouncing to the chiral character of the theory.

Finally, the calculation for the case of fermions in the (anti) fundamental representation goes along similar lines as the one outlined above. In this case the net noncommutative phase in the diagram is equal to one, and the anomaly can be obtained by replacing the group theoretical factor in (4.9) with $2\,\mathrm{tr}\,(T^a T^b) = \delta^{ab}$. Again for the U(1) case it is convenient to change the normalization to $T^0 = 1$. Then the resulting anomaly is half the value found for the adjoint U(1) case.

5. Summary and discussion

In this paper, we have studied noncommutative chiral gauge theories on the lattice by considering a chiral fermionic action containing a Dirac operator satisfying the Ginsparg-Wilson relation. When the fermions are in the adjoint representation of the gauge group $\mathrm{U}(r)$ we found that one can construct chiral gauge theories on the lattice with manifest star-gauge invariance in arbitrary even dimensions. In the continuum, it was known that the gauge anomaly cancels for the adjoint representation in four dimensions. However, if we do the same continuum calculation in two dimensions, the gauge anomaly remains. Therefore, our results indicate a certain dependence on the regularization procedure.

This is quite reminiscent of the situation in commutative space-time with odd dimensions. There the parity anomaly conflicts with the gauge symmetry in some cases [25]. If one imposes the gauge invariance, one obtains the parity anomaly. If one imposes parity invariance, one obtains the gauge anomaly. The results depend on regularization schemes. But the results are related to each other by adding an appropriate local counterterm, which is nothing but a Chern-Simons term. It is therefore suggested that, in the case at hand, the gauge anomaly obtained in the continuum calculation might be cancelled by adding some counterterm as well. Although the noncommutativity of the space-time inevitably introduces non-locality, we speculate that the counterterm is written in such a way that the non-locality is restricted to that encoded in the star-product. It would be interesting to identify the explicit form of the counterterm and we leave it for future investigations.

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